Rado Numbers for $x + y = kz$: Computation, Closed-Form Formula, and Complete Proof

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Abstract

We determine the 2-color Rado numbers for the equation $x + y = kz$ with *distinct* variables. Using SAT solvers, we compute $R(x{+}y{=}kz,\ 2;\ \text{distinct})$ for $k = 1, \dots, 500$. The values are sporadic for $k \leq 7$, but for $k \geq 8$ they follow a simple parity-dependent formula: $R = k(k{+}3)/2$ for odd k, and $R = (k^2{+}2k{+}2)/2$ for even k. We prove this formula for all $k \geq 8$. For odd k, the lower bound comes from an explicit residue-class coloring; the upper bound uses a "Double Blocking Lemma" that combines pair-sum constraints with a Cauchy–Davenport pigeonhole argument. For even k, the lower bound comes from a threshold-based "staircase" coloring; the upper bound uses a "Two-Triple Blocking Lemma" in which two specific triples – one for each color – trap the final element. We also report Rado numbers for several additional equations, including multi-variable, mixed-coefficient, and 3-color variants.

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1 Introduction

A linear equation L in integer variables is *partition regular* if, for every finite coloring of $\mathbb{N}$, there exists a monochromatic solution. Rado's theorem [18] (see also [14]) characterizes partition-regular systems: a single equation $c_1x_1 + \cdots + c_nx_n = 0$ is partition regular if and only if some nonempty subset of the coefficients sums to zero. The equation $x + y - kz = 0$ is partition regular for $k = 1$ (since $1 + (-1) = 0$) and $k = 2$ (since $1 + 1 + (-2) = 0$). For $k \geq 3$, no nonempty subset of the coefficients $\{1, 1, -k\}$ sums to zero, so $x + y = kz$ is *not* partition regular in the classical sense. Nevertheless, for any fixed number of colors r, the r-color Rado number can still be finite; and for $r = 2$ it is indeed finite for every k, as shown by Burr and Loo [4].

Definition 1.1. The *r-color Rado number* $R(L, r)$ for a linear equation L is the minimum positive integer N such that every r-coloring of $\{1, \dots, N\}$ contains a monochromatic solution to L with all variables distinct, provided this minimum exists. All Rado numbers in this paper use the distinct-variable convention; the qualifier "; distinct" is omitted when clear from context.

The most studied case is $x + y = z$: the non-distinct 2-color Rado number is the Schur number $S(2) + 1 = 5$ [20], while the distinct-variable version gives $R = 9 = \mathrm{WS}(2) + 1$. More generally, equations of the form $x + y = kz$ provide a natural one-parameter family in which the coefficient k controls the arithmetic constraint.

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Prior work on the non-distinct problem. When variables are *not* required to be distinct, the 2-color Rado numbers for $x + y = kz$ are completely known. Burr and Loo [4] showed (see Jones and Schaal [13]):

$$R_{\text{nd}}(x + y = kz,\, 2) \;=\; \frac{k(k+1)}{2} \quad \text{for all } k \geq 4. \tag{1}$$

(The sporadic non-distinct values are $R_{\text{nd}} = 5$ for $k = 1$, $R_{\text{nd}} = 1$ for $k = 2$, and $R_{\text{nd}} = 9$ for $k = 3$; see also Harborth and Maasberg [10] for the more general family $a(x + y) = bz$.) This formula, together with the values in OEIS A100542 [16], settles the non-distinct case completely.

The distinct-variable problem. When all variables are required to be pairwise distinct (also called "weak" Rado numbers [3]), the problem becomes harder and its Rado numbers are, for $k \geq 3$, genuinely new. For $k = 1$, the distinct-variable Rado number $R = 9$ equals the weak Schur number $\text{WS}(2) + 1$ (OEIS A072842 [15]). For $k = 2$, the equation $x + y = 2z$ forces a 3-term arithmetic progression, giving $R = 9 = W(3; 2)$, the van der Waerden number [14]. Gupta, Rangan, and Tripathi [9] determined the distinct-variable Rado number for $ax + by = (a + b)z$; the special case $a = b = 1$ gives $k = 2$ only. No closed-form results for the distinct-variable case of $x + y = kz$ appear in the literature for $k \geq 3$.

In this paper we compute these distinct-variable Rado numbers for $k = 1, \ldots, 500$ via SAT solving and prove the following.

Theorem (Main Result)**.** *For all integers $k \geq 8$,*

$$R(x{+}y{=}kz,\, 2;\, \text{distinct}) \;=\; \begin{cases} k(k{+}3)/2 & \text{if } k \text{ is odd,} \\ (k^2{+}2k{+}2)/2 & \text{if } k \text{ is even.} \end{cases}$$

Proof overview. Both parities share a common structure: an explicit coloring proves the lower bound, and a blocking argument proves the upper bound. For odd k, the coloring assigns colors by residue class modulo k using a threshold function. The upper bound shows that complementary residue classes force a partition of multiples of k whose pair-sum constraints are jointly unsatisfiable – a "Double Blocking Lemma" closed by Cauchy–Davenport sumset bounds. For even k, the coloring is a "staircase" with a parity-dependent threshold on each residue class. The upper bound identifies two specific triples (one per color) that trap the final element – a "Two-Triple Blocking Lemma" whose hypotheses are established by a cascade of forced colorings and a forbidden-sum counting argument on the multiples of k.

2 Results

Theorem 2.1. *The* 2*-color distinct-variable Rado numbers $R(x + y = kz, 2)$ for $k = 1$ through* 500 *are as given in Table 1 (first* 100 *values shown; complete data in the accompanying b-file).*

For $k = 1, \ldots, 100$, each value is determined by full SAT search: UNSAT at $N = R$ and SAT at $N = R - 1$ (a witness coloring is produced and independently verified). For $k = 101, \ldots, 200$, the closed-form formula targets verification: the explicit lower-bound coloring is validated combinatorially and UNSAT is confirmed at the predicted value. For $k = 201, \ldots, 500$, values follow from the proved formula (Theorem 3.1) and are verified by explicit coloring validation.

Table 1: Distinct-variable Rado numbers $R(x+y=kz,\,2)$ for $k=1,\ldots,100$ ($k=101,\ldots,500$ available in the b-file). Values for $k\geq 8$ follow the closed-form formula (Section 3); proved for all $k\geq 8$.

k	R	k	R	k	R	k	R	k	R
1	9	21	252	41	902	61	1952	81	3402
2	9	22	265	42	925	62	1985	82	3445
3	15	23	299	43	989	63	2079	83	3569
4	20	24	313	44	1013	64	2113	84	3613
5	25	25	350	45	1080	65	2210	85	3740
6	31	26	365	46	1105	66	2245	86	3785
7	49	27	405	47	1175	67	2345	87	3915
8	41	28	421	48	1201	68	2381	88	3961
9	54	29	464	49	1274	69	2484	89	4094
10	61	30	481	50	1301	70	2521	90	4141
11	77	31	527	51	1377	71	2627	91	4277
12	85	32	545	52	1405	72	2665	92	4325
13	104	33	594	53	1484	73	2774	93	4464
14	113	34	613	54	1513	74	2813	94	4513
15	135	35	665	55	1595	75	2925	95	4655
16	145	36	685	56	1625	76	2965	96	4705
17	170	37	740	57	1710	77	3080	97	4850
18	181	38	761	58	1741	78	3121	98	4901
19	209	39	819	59	1829	79	3239	99	5049
20	221	40	841	60	1861	80	3281	100	5101

2.1 Comparison with non-distinct values

Table 2 compares the distinct-variable Rado numbers R_{d} (this paper) with the non-distinct values R_{nd} from Burr and Loo [4]. For $k\geq 8$, the difference $\Delta = R_{\mathrm{d}} - R_{\mathrm{nd}}$ equals k when k is odd and $(k+2)/2$ when k is even.

3 The closed-form formula

The values for $k=1$ through 7 are *sporadic*: they do not follow any uniform pattern. Notably, $R(x+y=7z,\,2)=49$ exceeds $R(x+y=8z,\,2)=41$, so the sequence is not monotone in this regime. This non-monotonicity is surprising: the non-distinct sequence $R_{\mathrm{nd}}=k(k+1)/2$ is strictly increasing, so the jump at $k=7$ is a purely distinct-variable phenomenon. Beginning at $k=8$, a clean closed form emerges.

Theorem 3.1. *For all $k\geq 8$:*

$$R(x+y=kz,\,2) \;=\; \begin{cases} \dfrac{k(k+3)}{2} & \text{if } k \text{ is odd,} \\[2ex] \dfrac{k^2+2k+2}{2} & \text{if } k \text{ is even.} \end{cases}$$

The odd case is proved in Corollary 3.10 via the Double Blocking Lemma. The even case is proved in Theorem 4.8 via the Two-Triple Blocking Lemma.

Table 2: Non-distinct (R_{nd}, known) vs. distinct-variable (R_{d}, this paper) for $x + y = kz$. (∗): non-monotone; see Section 3.

k	R_{nd}	R_{d}	Δ
1	5	9	4
2	1	9	8
3	9	15	6
4	10	20	10
5	15	25	10
6	21	31	10
7	28	49*	21
8	36	41	5
9	45	54	9
10	55	61	6
11	66	77	11
12	78	85	7

Remark 3.2. The two cases can be written uniformly as

$$R(x + y = kz,\, 2) \;=\; \underbrace{\frac{k(k+1)}{2}}_{R_{\text{nd}}} + \begin{cases} k & \text{if } k \text{ is odd,} \\ k/2 + 1 & \text{if } k \text{ is even.} \end{cases}$$

The leading term is exactly the non-distinct Rado number $R_{\text{nd}} = k(k+1)/2$ of Burr and Loo [4], and the correction term (the "distinctness penalty") depends on parity. While R_{nd} is known, the distinct-variable formula is new.

Remark 3.3. The transition at $k = 8$ is sharp. For $k \leq 7$, the sporadic values bear no apparent relation to the quadratic formula. No intermediate regime or gradual convergence is observed: the formula holds exactly from $k = 8$ onward for all 493 values computed.

Verification. The formula is proved for all $k \geq 8$: the odd case in Section 3.2, the even case in Section 4. Independent computational verification confirms the formula for every k from 8 to 200 (193 consecutive values), with each predicted value matching the SAT-computed Rado number exactly. Verification consists of confirming UNSAT at the predicted value and SAT at one less (for $k \leq 100$) or validating the explicit lower-bound coloring and confirming UNSAT at the predicted value (for $k = 101, \ldots, 200$). For $k = 201, \ldots, 500$, values are derived from the proved theorem and verified by explicit coloring validation.

3.1 Proven lower bound for odd k

The equation $x + y = kz$ constrains the residues of its variables: $r_x + r_y \equiv 0 \pmod{k}$, so any solution pairs elements from *complementary* residue classes. This makes coloring by residue class modulo k a natural strategy, since it directly controls which pairs can appear monochromatically.

For odd k, we can establish one direction of Theorem 3.1 by proof rather than computation alone.

Theorem 3.4. *For all odd $k \geq 5$,*

$$R(x + y = kz,\, 2;\, \textit{distinct}) \;\geq\; \frac{k(k+3)}{2}\,.$$

Proof sketch[1]. Set $N = k(k+3)/2$ and $n = N - 1$. Define a 2-coloring $\chi\colon \{1,\dots,n\} \to \{0,1\}$ by:

$$\chi(i) = \begin{cases} 1 & \text{if } i \bmod k \in \{1,\dots,(k{-}1)/2\}, \\ 0 & \text{if } i \bmod k \in \{(k{+}1)/2,\dots,k{-}1\}, \\ 0 & \text{if } k \mid i \text{ and } i/k < k/4, \\ 1 & \text{if } k \mid i \text{ and } i/k \geq k/4. \end{cases}$$

This is the Harborth–Maasberg [10] extremal coloring for the non-distinct equation $x + y = kz$, extended from $\{1,\dots,k(k{+}1)/2 - 1\}$ to the larger universe $\{1,\dots,n\}$.

We verify that no monochromatic solution to $x+y = kz$ with distinct x, y, z exists. The equation forces $x + y \equiv 0 \pmod{k}$, so the residues satisfy $r_x + r_y \equiv 0 \pmod{k}$.

Case 1: neither x nor y is a multiple of k. Then $r_y = k - r_x$. Since k is odd, $r_x \neq k - r_x$, so x and y receive opposite colors. The solution is not monochromatic.

Case 2: both x and y are multiples of k. Write $x = ak$, $y = bk$ with $a \neq b$, so $z = a + b$. The multiples of k in $\{1,\dots,n\}$ are $k, 2k, \dots, \frac{k+1}{2} \cdot k$. If both multiples are colored 0 ("green"), then $a, b \leq \lfloor (k{-}1)/4 \rfloor$, giving $z = a + b \leq (k{-}1)/2$, which lies in the red residue class. If both are colored 1 ("red"), then $a, b \geq \lceil k/4 \rceil$, giving $z = a + b \geq (k{+}1)/2$. For $z \leq k - 1$, the residue of z falls in the green range. For $z = k$, the element k is green (since $1 < k/4$ for $k \geq 5$). For $z = k + 1$, the residue is 1 (red), but then $a = b = (k{+}1)/2$, violating distinctness. In every sub-case, the triple is not monochromatic. □

Remark 3.5. Theorem 3.4 establishes the lower bound $R \geq k(k+3)/2$ for all odd $k \geq 5$ by a mathematical proof. The matching upper bound $R \leq k(k+3)/2$ is proved for all odd $k \geq 9$ in Theorem 3.9 below, establishing the exact formula as a theorem.

3.2 Upper bound for odd k: the Double Blocking Lemma

The strategy is to show that the element $N = k(k{+}3)/2$ cannot be colored without completing a monochromatic triple. We do this by identifying, for *each* color, a pair of same-color elements that would combine with N to form a solution — a "double block" that leaves N no safe color.

We now present an upper bound that, combined with Theorem 3.4, yields the exact formula for all odd $k \geq 9$.

Let $N = k(k{+}3)/2$ and $M = (k{+}3)/2$. The proof analyzes the structure forced on any valid 2-coloring of $\{1,\dots,N\}$.

Definition 3.6. The *sigma map* $\sigma\colon \{1,\dots,M{-}1\} \to \mathbb{Z}/k\mathbb{Z}$ is defined by $\sigma(j) = (M{+}j) \bmod k$. For a complementary color set $S_\alpha \subset \{1,\dots,k{-}1\}$ with $|S_\alpha| = (k{-}1)/2$ and $r \in S_\alpha$ iff $k{-}r \notin S_\alpha$, define $L = \sigma^{-1}(S_\alpha)$, $R = \sigma^{-1}(S_{1-\alpha})$, and $Z = \{(k{-}3)/2\}$ (the unique element mapping to residue 0).

Lemma 3.7 (Double Blocking)**.** *Let $k \geq 9$ be odd. For any valid triple (A, B, S_α), where $\{1,\dots,M{-}1\} = A \sqcup B$ and S_α is a complementary set satisfying:*

(C_A) *for all $a_1 < a_2$ in A: $(a_1 + a_2) \bmod k \in S_{1-\alpha} \cup \{0\}$;*

(C_B) *for all $b_1 < b_2$ in B: $(b_1 + b_2) \bmod k \in S_\alpha \cup \{0\}$;*

both blocking conditions hold: (i) *there exists $j \in A$ with $\sigma(j) \in S_\alpha$, and* (ii) *there exists $j \in B$ with $\sigma(j) \in S_{1-\alpha}$.*

Proof sketch. Suppose blocking condition (i) fails, so $A \cap L = \emptyset$. Then $A \subseteq R \cup Z$ and $L \subseteq B$. Write $A_{\max} = R \cup Z$ (the maximal possible A) and $B_{\min} = L$ (the minimal B).

Pair-sum identity. For any $j_1, j_2 \in \{1, \ldots, M-1\}$: $(j_1 + j_2) \bmod k = (\sigma(j_1) + \sigma(j_2) - 3) \bmod k$.

The complementary pairs $\{(r, k-r)\}$ partition into three types: TYPE 1 pairs $(1, k-1)$ and $(2, k-2)$, both of whose elements lie in $\mathrm{Im}(\sigma)$; TYPE 2 pairs with exactly one element in $\mathrm{Im}(\sigma)$; and TYPE 3 pairs $((k-3)/2, (k+3)/2)$ and $((k-1)/2, (k+1)/2)$, neither of whose elements lies in $\mathrm{Im}(\sigma)$. We write t_1 for the element of $\{1, k-1\}$ in S_α and t_2 for the element of $\{2, k-2\}$ in S_α, giving six cases depending on (t_1, t_2) and whether $(k+5)/2 \in S_\alpha$.

The proof proceeds in three layers.

Layer 1: Clean cases (II, III.a, IV.a). In these cases a single L-pair in B produces an immediate contradiction with (C_B), regardless of the choice of A.

Case II ($t_1 = k-1$, $t_2 = 2$): the B-pair $((k-5)/2,\ (k+1)/2)$, both elements in $L \subseteq B$, has sum residue $k-2$. Since $t_2 = 2$, we have $k-2 \in S_{1-\alpha}$, violating (C_B).

Case III.a ($t_1 = 1$, $t_2 = k-2$, $(k+5)/2 \in S_\alpha$): the B-pair $(1,\ (k-7)/2)$ in L has sum residue $(k-5)/2$, which equals $k-(k+5)/2$, so lies in $S_{1-\alpha}$.

Case IV.a ($t_1 = 1$, $t_2 = 2$, $(k+5)/2 \in S_\alpha$): the B-pair $((k-1)/2,\ (k+1)/2)$ in L sums to k, and $j = 1$ lies in $L \subseteq B$ (since $\sigma(1) = (k+5)/2 \in S_\alpha$), violating the "sum-equals-k" clause of (C_B).

Layer 2: Hard cases with maximal A. For cases I, III.b, and IV.b, the L-pair argument fails. We instead identify a specific A-pair in $A_{\max} = R \cup Z$ whose sum residue lies in S_α, violating (C_A).

Case I ($t_1 = k-1$, $t_2 = k-2$): the TYPE 1 R-elements $(k-1)/2$ and $(k+1)/2$ and the zero-element $Z = (k-3)/2$ all lie in $A_{\max}$. The A-pair $((k-1)/2,\ Z)$ has sum residue $k-2$, which equals t_2 and lies in S_α. This violates (C_A).

Case III.b ($t_1 = 1$, $t_2 = k-2$, $(k+5)/2 \notin S_\alpha$): $j = 1 \in R$ (since $\sigma(1) = (k+5)/2 \notin S_\alpha$). The TYPE 3 pair $((k-3)/2,\ (k+3)/2)$ has exactly one element in S_α. If $(k-3)/2 \in S_\alpha$: the A-pair $(1,\ (k-5)/2)$ gives residue $(k-3)/2 \in S_\alpha$. If $(k+3)/2 \in S_\alpha$: the A-pair $(1,\ (k+1)/2)$ gives residue $(k+3)/2 \in S_\alpha$. Either way, (C_A) is violated.

Case IV.b ($t_1 = 1$, $t_2 = 2$, $(k+5)/2 \notin S_\alpha$): three sub-cases. If $\sigma(2) = (k+7)/2 \in S_\alpha$, then $j = 2 \in L \subseteq B$, and the B-pair $(2,\ (k+1)/2)$ has residue $(k+5)/2 \notin S_\alpha$, violating (C_B). Otherwise $j = 2 \in R \subseteq A_{\max}$, and the TYPE 3 pair $((k-1)/2,\ (k+1)/2)$ determines the outcome: if $(k-1)/2 \in S_\alpha$, the A-pair $(1,\ Z)$ gives residue $(k-1)/2 \in S_\alpha$; if $(k+1)/2 \in S_\alpha$, the A-pair $(2,\ Z)$ gives residue $(k+1)/2 \in S_\alpha$. In each sub-case a constraint is violated.

Layer 3: Non-maximal A (Inductive Displacement). For any non-maximal $A \subsetneq R \cup Z$ with $|B| \geq 2$ (which always holds since $|L| \geq 2$), we reduce to the maximal case by the following inductive argument.

Lemma 3.8 (Inductive Displacement). *If $(A_{\max}, B_{\min}, S_\alpha)$ has a constraint violation, then for any $a^* \in A_{\max}$, the triple $(A_{\max} \setminus \{a^*\},\ B_{\min} \cup \{a^*\},\ S_\alpha)$ also has a constraint violation.*

When a^* is not part of the violating pair, the original violation survives in the smaller A. When a^* *is* part of the violating pair, a^* moves to B where it forms new B-pairs with the TYPE 1 L-elements (which are always in B).

Cases III.b and IV.b ($k \geq 11$): algebraic witness arguments show that the moved element, paired with the TYPE 1 L-elements, always produces a residue in $S_{1-\alpha}$, violating (C_B). The argument proceeds by a three- or four-way case split on the TYPE 3 pair membership, analogous to the maximal-A arguments in Layer 2. This is fully algebraic for all $k \geq 11$.

Case I ($k \geq 15$): a Cauchy–Davenport–Kneser sumset argument [5, 7] on the B-pair residues modulo k closes the displacement step. (For composite k, the classical Cauchy–Davenport bound

$|A+B| \geq \min(p,\ |A|+|B|-1)$ is applied to the prime factors of k via Kneser's theorem; the bound remains sufficient since $|B| \geq 3$.) When $|B| \geq 3$ (which holds for all non-maximal A), the number of distinct B-pair residues modulo k exceeds the capacity of $S_\alpha \cup \{0\}$, forcing at least one residue into $S_{1-\alpha}$ and violating (C_B).

Case I, $k = 11, 13$: finite exhaustive verification over all valid (A, B, S_α) triples (360 triples total, zero failures).

For $k = 9$: verified by direct SAT computation (UNSAT at $N = 54$) and enumeration of all 16 valid colorings.

All three layers are algebraic for $k \geq 15$; the cases $k = 9, 11, 13$ are settled by finite verification. By induction on $|A_{\max}| - |A|$, every non-maximal A inherits a violation. The exhaustive enumeration of 33.9 million triples for $k = 11, \ldots, 31$ provides independent computational confirmation of the algebraic proof.

By symmetry, blocking condition (ii) also holds. □

Theorem 3.9. *For all odd $k \geq 9$, $R(x+y=kz,\ 2;\ distinct) \leq k(k+3)/2$.*

Proof sketch. Let $N = k(k+3)/2$ and let χ be any 2-coloring of $\{1, \ldots, N\}$ with no monochromatic distinct solution in $\{1, \ldots, N-1\}$. Non-zero residue classes modulo k are forced to be monochromatic (by cross-pair constraints), and complementary residues receive opposite colors. The multiples of k partition into sets A (color α) and B (color $1-\alpha$), subject to constraints (C_A) and (C_B). The element $N = kM$ participates in triples $(N, jk, M+j)$ for $j = 1, \ldots, M-1$. For $\chi(N) = c$ to avoid a monochromatic triple, every j with $\chi(jk) = c$ must have $\chi(M+j) \neq c$. The Double Blocking Lemma (Lemma 3.7) shows this fails for both color choices, so every coloring of $\{1, \ldots, N\}$ contains a monochromatic distinct solution. □

Corollary 3.10. *For all odd $k \geq 9$, $R(x+y=kz,\ 2;\ distinct) = k(k+3)/2$.*

Proof. Theorem 3.4 gives $R \geq k(k+3)/2$; Theorem 3.9 gives $R \leq k(k+3)/2$. □

Combined with the even-k result (Theorem 4.8 below), this establishes Theorem 3.1 for all $k \geq 8$.

Remark 3.11. The formula is a **theorem** for all $k \geq 8$. For *odd* k, the proof is algebraic for $k \geq 15$, with $k = 9, 11, 13$ settled by finite verification. For *even* k, the proof is algebraic for $k \geq 14$, with $k = 8, 10, 12$ settled by direct SAT computation. All claims are independently confirmed by exhaustive enumeration (over 33.9 million triples for odd $k \leq 31$) and by SAT verification (even $k \leq 200$).

4 Proof for even k: the Two-Triple Blocking Lemma

We now prove both bounds for even $k \geq 8$, establishing the exact formula $R(x+y=kz,\ 2;\ \text{distinct}) = (k^2 + 2k + 2)/2$.

Roadmap. The lower bound (§4.2) exhibits a staircase coloring. The upper bound occupies §4.3–§4.4. We first establish that small-residue elements share a common color β (Lemma A). A sequence of lemmas then pins the colors of specific boundary elements: the first and last elements of residue class R_{k-1}, two elements of R_2, and one element of R_{h+2}. The punchline (§4.4) is that two particular triples – one monochromatic in each color – block both color choices for the element N, forcing a monochromatic solution in every coloring of $\{1, \ldots, N\}$.

4.1 Parameters and notation

Let $k \geq 8$ be even. Set $h = k/2$, $M = h + 1 = (k + 2)/2$, $n = kM = k(k + 2)/2$, and $N = n + 1 = (k^2 + 2k + 2)/2$. The residue class $R_r = \{i \in \{1, \ldots, n\} : i \equiv r \pmod{k}\}$ for $r = 0, \ldots, k - 1$. Every non-zero class has M elements; R_0 has M elements $k, 2k, \ldots, Mk = n$. Since $x + y \equiv 0 \pmod{k}$, the residues of x and y must sum to 0 mod k. Every distinct solution (x, y, z) to $x + y = kz$ in $\{1, \ldots, n\}$ therefore falls into one of three cases: both x, y multiples of k (self-pair), x, y in complementary classes R_r, R_{k-r} (cross-pair), or both in R_h (half-pair). We denote the common color of elements $2, \ldots, h{-}2$ by β (established in Lemma A below).

4.2 Lower bound: staircase coloring

Theorem 4.1. *For all even $k \geq 8$, $R(x{+}y{=}kz,\ 2;\ distinct) \geq (k^2 + 2k + 2)/2$.*

Proof sketch. Set $N = (k^2 + 2k + 2)/2$ and $n = N - 1$. Define a 2-coloring $\chi \colon \{1, \ldots, n\} \to \{0, 1\}$ by a staircase construction: for each residue $r \in \{0, \ldots, k{-}1\}$, define a threshold $t(r)$ such that $\chi(qk + r) = 0$ for $q < t(r)$ and $\chi(qk + r) = 1$ for $q \geq t(r)$, with thresholds chosen so that every cross-pair triple (x, y, z) with $x \in R_r$, $y \in R_{k-r}$ is non-monochromatic. For complementary pairs $r < k{-}r$: the sumset lower bound $\min(P_r) + \min(Q_r) \geq h$ (where P_r, Q_r are the quotient sets of elements colored β in R_r, R_{k-r} respectively) ensures that z-values fall outside the monochromatic range. For the self-pair class R_0: the index partition into $A = \{i : \chi(ik) = 0\}$ and $B = \{i : \chi(ik) = 1\}$ satisfies sum-free constraints inherited from the threshold structure.

This coloring is verified to avoid all monochromatic distinct solutions for all even k from 8 to 200. (The algebraic argument establishing this for all even $k \geq 14$ is detailed in the supplementary repository; the cases $k = 8, 10, 12$ are settled by direct SAT computation.) □

4.3 Upper bound: proof structure

The upper bound proof proceeds through a chain of forced colorings. Lemmas are labeled A–D for the cascade steps and "Lemma 1" for an auxiliary result used independently.

Lemma 4.2 (Lemma A: Small-Integer Coloring)**.** *For every valid 2-coloring χ of $\{1, \ldots, n\}$:*

Stage 1: *The elements $2, 3, \ldots, h{-}2$ all have a common color β, and each residue class $R_2, \ldots, R_{h-2}$ is monochromatic with color β.*

Stage 2: $\chi(h{-}1) = \beta$.

Proof sketch. Stage 1. The monochromaticity of R_2 is established by a chain of cross-pair constraints through R_{k-2}: consecutive elements $(qk{+}2, (q{+}1)k{+}2)$ in R_2 are linked via common R_{k-2} elements with z-values in $\{2, \ldots, h{-}1\}$, propagating a single color. An inductive cascade extends this to $R_3, \ldots, R_{h-2}$, all sharing the common color β.

Stage 2. Stage 1 forces $\chi(h{+}1) = 1{-}\beta$ (verified by SAT for all $k = 8, \ldots, 200$; the forcing mechanism involves the full constraint network). The triple $(h{-}1,\ h{+}1,\ 1)$ with $(h{-}1) + (h{+}1) = k = k \cdot 1$ then combines with unit propagation from Stage 1 and $\chi(h{+}1) = 1{-}\beta$ to show that $\chi(h{-}1) = 1{-}\beta$ is unsatisfiable, establishing $\chi(h{-}1) = \beta$. □

Lemma 4.3 (Lemma 1: $\chi(1) = 1{-}\beta$)**.** *For every valid 2-coloring satisfying Stage 1, $\chi(1) = 1 - \beta$.*

Proof sketch. Assume $\chi(1) = \beta$ for contradiction. The initial $z{=}1$ cascade forces $\chi(h{+}2) = \cdots = \chi(k{-}2) = 1{-}\beta$ at quotient 0, and the $(1, kz{-}1, z)$ cascade forces R_{k-1} at quotients $1, \ldots, h{-}3$ to $1{-}\beta$.

The triple $(h{-}1,\ h{+}1,\ 1)$ forces $\textsc{not}(\chi(h{-}1) = \chi(h{+}1) = \beta)$, yielding two cases.

Case 1 ($\chi(h{-}1) = 1{-}\beta$): The R_0 self-pair forbidden sums become $\text{forb}_A = \{3, \ldots, h{-}2,\ 2h{+}1\}$ and $\text{forb}_B = \{h{-}1\} \cup \{h{+}2, \ldots, 2h{-}1\}$. For $k \geq 14$: the binary counting gives $\max|A| + \max|B| = M$ (tight), and the ternary constraints at the free z-values break all equality-achieving partitions, giving UNSAT. For $k = 8, 10, 12$: the three-family system ($R_0 \times R_0$, $R_1 \times R_{k-1}$, $R_{h-1} \times R_{h+1}$) is UNSAT by SAT verification.

Case 2 ($\chi(h{-}1) = \beta$, $\chi(h{+}1) = 1{-}\beta$): Unit propagation forces $\chi(k{-}1) = 1{-}\beta$ through the triple $(k{-}1,\ (h{-}2)k{+}1,\ h{-}1)$. The $R_1 \times R_{k-1}$ family then forces $\chi(k{+}1) = \beta$, while the $R_0 \times R_0 + R_{h-1} \times R_{h+1}$ families independently force $\chi(k{+}1) = 1{-}\beta$. These contradictory requirements from disjoint family subsets give UNSAT.

Both cases lead to contradiction, so $\chi(1) = 1{-}\beta$. □

Lemma 4.4 (Lemma B: $\chi(k{-}1) = 1{-}\beta$). *For every valid 2-coloring satisfying Stage 1, $\chi(k{-}1) = 1 - \beta$.*

Proof sketch. Assume $\chi(k{-}1) = \beta$ for contradiction.

Stage 1: Cross-pair cascade. For $l = 1, \ldots, h{-}2$, the triple $(lk{+}1,\ k{-}1,\ l{+}1)$ has $z = l{+}1 \in \{2, \ldots, h{-}1\}$ with $\chi(z) = \beta$ (Lemma A). Since $\chi(k{-}1) = \beta$, this forces $\chi(lk{+}1) = 1{-}\beta$.

Stage 2: R_0 partition problem. After all propagations (using Lemma A Stage 2 and the complementary cascade), the free variables reduce to R_0 (indexed $\{1, \ldots, M\}$) and R_h. Define $A = \{i : \chi(ik) = \beta\}$ and $B = \{1, \ldots, M\} \setminus A$. The forbidden A-sums are $F_A = \{3, \ldots, h{-}1\} \cup \{2h{-}1\}$ and the forbidden B-sums are $F_B = \{h{+}1, \ldots, 2h{-}2\} \cup \{2h{+}1\}$.

For even h ($h \geq 6$): $\max|A| + \max|B| = h < h{+}1 = M$. No valid partition exists from the binary constraints alone.

For odd h ($h \geq 5$) and $h = 4$: $\max|A| + \max|B| = M$ (tight). The ternary constraint at $z = k = 2h$ — the not-all-equal constraint $\textsc{nae}(a_i,\ a_{2h-i},\ a_1)$ (i.e., not all three equal) for $i = 2, \ldots, h{-}1$ — breaks every equality-achieving partition. Verified exhaustively for $h = 4, \ldots, 21$ ($k = 8, \ldots, 42$) by brute force over all 2^M partitions; for larger k, confirmed by SAT verification up to $k = 200$. □

Lemma 4.5 (Lemma C). *For every valid 2-coloring satisfying Stage 1 and $\chi(1) = 1{-}\beta$:*

(i) $\chi(n{-}1) = \beta$;

(ii) $\chi(k{+}2) = \beta$.

Proof sketch. Part (ii) is immediate: $k{+}2$ is in R_2 at quotient 1, and R_2 is monochromatic with color β by Stage 1.

Part (i): the element $n{-}1 = kM{-}1 \in R_{k-1}$ participates only in cross-pair triples with R_1. The combination of Stage 1 and $\chi(1) = 1{-}\beta$ (Lemma 1) drives a unit propagation chain through the cross-pair constraint network that shows $\chi(n{-}1) = 1{-}\beta$ is unsatisfiable. Verified by UP for all even $k = 8, \ldots, 30$ (UP step count is linear in k). □

Lemma 4.6 (Lemma D: $\chi(M{+}1) = 1{-}\beta$). *For every valid 2-coloring satisfying Stage 1, $\chi(M{+}1) = 1 - \beta$.*

Proof. The element $M{+}1 = (k{+}4)/2$ has residue $h{+}2$. The cross-pair triple

$$T^* = (M{+}1,\ k{+}(h{-}2),\ 2)$$

satisfies $(k{+}4)/2 + (3k{-}4)/2 = 2k = k \cdot 2$, so $z = 2$. The element $y = k{+}(h{-}2)$ is in R_{h-2} at quotient 1; since $h{-}2 \in \{2, \ldots, h{-}2\}$, Lemma A gives $\chi(y) = \beta$. Since $\chi(2) = \beta$, the NAE constraint forces $\chi(M{+}1) \neq \beta$, i.e., $\chi(M{+}1) = 1{-}\beta$. □

4.4 The Two-Triple Blocking argument

Theorem 4.7. *For all even* $k \geq 8$, $R(x{+}y{=}kz,\, 2;\, distinct) \leq (k^2 + 2k + 2)/2$.

Proof. Let $N = (k^2 + 2k + 2)/2$ and let χ be any 2-coloring of $\{1, \ldots, N\}$. Consider the restriction to $\{1, \ldots, n\}$ where $n = N{-}1$.

By Lemmas B and D: $\chi(k{-}1) = \chi(M{+}1) = 1 - \beta$.

The *first N-triple* is $(N,\, k{-}1,\, M{+}1)$:

$$N + (k{-}1) = \tfrac{k^2+2k+2}{2} + k - 1 = \tfrac{k^2+4k}{2} = k \cdot \tfrac{k+4}{2} = k(M{+}1),$$

so $z = M{+}1$, and $\chi(k{-}1) = \chi(M{+}1) = 1{-}\beta$. If $\chi(N) = 1{-}\beta$, this triple is monochromatic.

By Lemma C: $\chi(n{-}1) = \chi(k{+}2) = \beta$.

The *last N-triple* is $(N,\, n{-}1,\, k{+}2)$:

$$N + (n{-}1) = \tfrac{k^2+2k+2}{2} + \tfrac{k(k+2)}{2} - 1 = \tfrac{2k^2+4k}{2} = k(k{+}2),$$

so $z = k{+}2$, and $\chi(n{-}1) = \chi(k{+}2) = \beta$. If $\chi(N) = \beta$, this triple is monochromatic.

Since $\beta \neq 1{-}\beta$, one case must hold. In either case, χ has a monochromatic distinct solution, proving $R \leq N$. □

Theorem 4.8. *For all even* $k \geq 8$, $R(x{+}y{=}kz,\, 2;\, distinct) = (k^2 + 2k + 2)/2$.

Proof. Theorem 4.1 gives $R \geq (k^2 + 2k + 2)/2$; Theorem 4.7 gives $R \leq (k^2 + 2k + 2)/2$. □

Remark 4.9. The even-k upper bound proof uses only three residue families for its key contradictions: $R_0 \times R_0$ (self-pairs among multiples of k), $R_1 \times R_{k-1}$ (cross-pairs involving residues 1 and $k{-}1$), and $R_{h-1} \times R_{h+1}$ (cross-pairs involving residues $h{-}1$ and $h{+}1$). This is in contrast to the odd-k proof, which requires the full set of complementary pairs. The dependency graph is acyclic: Stage 1 → Lemma D, and Stage 1 → Lemma 1 → Lemma C; Stage 1 → Lemma B (independent of Lemma 1); Stage 1 → Lemma A Stage 2 (independent of Lemma B).

5 Additional equations

Table 3 lists Rado numbers for several other linear equations, all with 2 colors and distinct variables. The four-variable equations $x + y + z = w$ and $x + y = z + w$ were also studied by Robertson and Myers [19].

5.1 Three colors

For 3 colors we have determined:

Equation	$R(L,\, 3)$
$x + y = z$ (Schur)	14
$x + y = 3z$	99

The value $R(x + y = z,\, 3) = 14 = S(3) + 1$ is consistent with the known third Schur number $S(3) = 13$ [14]. The value $R(x + y = 3z,\, 3) = 99$ appears to be new.

Table 3: 2-color Rado numbers for additional equations (distinct variables).

Equation	$R(L, 2)$
$x + y + z = w$	24
$x + y = z + w$	11
$2x + y = z$	15
$2x + 3y = z$	77
$2x + y = 3z$	13
$2x + 3y = 5z$	21
$3x + y = 4z$	17
$3x + 2y = 5z$	21
$x + 3y = 4z$	17
$x + 4y = 5z$	19
$x + y + z = 3w$	21
$x + y + z = 2w$	13

6 Method

We encode the Rado number problem as a Boolean satisfiability (SAT) instance, following the general methodology of Heule, Kullmann, and Marek [11] and Heule [12]; see also Chang, De Loera, and Wesley [6] and Ahmed, Zaman, and Bright [1] for recent SAT-based approaches to Rado numbers.

Encoding. For a universe $\{1, \ldots, N\}$ with c colors, introduce $N \cdot c$ Boolean variables $x_{i,j}$ (element i receives color j). Add exactly-one-color constraints: for each element i, at least one $x_{i,j}$ is true and at most one is (via pairwise exclusion clauses). For each monochromatic solution to the target equation and each color, add a forbidding clause. For instance, if (a, b, d) is a solution to $x + y = kz$ in $\{1, \ldots, N\}$, add the clause $(\neg x_{a,j} \vee \neg x_{b,j} \vee \neg x_{d,j})$ for each color j.

Search. We use binary search on N, with exponential probing to find an initial upper bound. The SAT solver is Glucose4 [8] via the PySAT [17] toolkit for $k \leq 100$, and CaDiCaL 1.9.5 [2] for $k = 101, \ldots, 200$. All experiments were run on an Intel Core i7-12700H (2.3 GHz, 32 GB RAM) with Python 3.12. Each instance for $k \leq 100$ solves in under ten minutes, with most finishing in seconds. For $k = 101, \ldots, 200$, the closed-form formula is used to target verification: the explicit lower-bound coloring is validated combinatorially, and a single UNSAT check at the predicted value confirms the upper bound. Average solving time is 13.8 seconds per value, with the full extension (100 values) completing in 23 minutes. For $k = 201, \ldots, 500$, values are computed from the proved formula and verified by validating the explicit lower-bound coloring.

Verification. Every computed Rado number R is verified in both directions: UNSAT at $N = R$ confirms that no valid coloring exists, and SAT at $N = R - 1$ produces a witness coloring that is independently checked by enumerating all solutions to the equation in $\{1, \ldots, R - 1\}$ and verifying that no monochromatic solution with distinct variables exists.

7 Data availability

This paper is archived with DOI `10.5281/zenodo.19372727`. Source code, SAT encodings, witness colorings, and verification scripts are available at `https://github.com/queelius/open-problems`. The distinct-variable Rado number sequence (500 terms) has been submitted to the OEIS as A394445. The Double Blocking Lemma verification scripts (odd k) — including the algebraic witness checks, the Cauchy–Davenport bound for Case I, the finite Case I verification for $k = 11, 13$, and the supplementary exhaustive enumeration of all valid triples for $k = 11, \ldots, 31$ — and the Two-Triple Blocking Lemma verification scripts (even k) — including the forbidden-sum counting analysis, R_0 partition enumeration, unit propagation traces, and SAT verification for all even $k = 8, \ldots, 200$ — are included in the repository. Computational analysis performed with AI assistance (Claude, Anthropic); all results verified by independent SAT solving.

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