The Latent/Observed Duality: A Unified Theory of Approximate Computing

Alexander Towell
Southern Illinois University Edwardsville
atowell@siue.edu

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Abstract

We present a unified type-theoretic framework for approximate computing based on the fundamental distinction between latent (true) and observed (approximate) values. This duality naturally arises not only in probabilistic data structures like Bloom filters, but equally in algorithms themselves—from primality testing to optimization heuristics. We formalize this concept through Bernoulli types—a family of probabilistic types parameterized by false positive and false negative rates—and show how they compose through algebraic data types. Our framework reveals that algorithms and data structures are dual: algorithms are values in function types that observe mathematical truth through computation, just as Bloom filters observe set membership. Our framework provides: (1) a systematic way to reason about error propagation in both data structures and algorithms, (2) a type-safe interface for probabilistic computations, and (3) a foundation for understanding the inherent trade-offs between space, time, and accuracy. We demonstrate the framework's utility through parallel examples of Bloom filters (data structure) and Miller-Rabin primality testing (algorithm), showing how both exhibit identical confusion matrix structures. The framework has been implemented as a header-only C++ library and validated on real-world applications.

1 Introduction

Modern computing systems increasingly rely on approximate methods to achieve scalability and efficiency. From probabilistic data structures like Bloom filters to approximate query processing in databases, these methods trade perfect accuracy for substantial gains in space and time complexity. However, reasoning about the composition and correctness of approximate computations remains challenging, particularly when errors compound through multiple operations.

We propose a unified framework based on a fundamental observation: approximate computing naturally involves two distinct types of values—latent values that represent true mathematical objects, and observed values that represent our noisy approximations of them. This latent/observed duality appears across diverse domains:

- In Bloom filters, the latent set is the true membership, while the observed set includes false positives
- In differential privacy, latent data is the true dataset, while observed data includes calibrated noise
- In sketching algorithms, latent distributions are approximated by observed frequency estimates

1.1 Contributions

This paper makes the following contributions:

1. **Type-theoretic framework**: We formalize the latent/observed duality through Bernoulli types, which explicitly model approximation errors as type parameters (Section 2).
2. **Composition theory**: We show how Bernoulli types compose through algebraic data types, providing systematic error propagation rules for products, sums, and recursive types (Section 3).
3. **Confusion matrix formalism**: We introduce a confusion matrix representation that unifies diverse probabilistic structures and enables reasoning about their algebraic properties (Section 4).
4. **Concrete constructions**: We demonstrate how classical probabilistic data structures emerge as instances of our framework, providing new insights into their design and analysis (Section 5).
5. **Implementation and evaluation**: We present a production-ready C++ implementation and evaluate its performance on real-world applications (Section 6).

1.2 Paper Organization

Section 2 introduces the core concepts of latent and observed types. Section 3 develops the composition theory for Bernoulli types. Section 4 presents the confusion matrix formalism. Section 5 shows concrete applications to Bloom filters and related structures. Section 6 describes our implementation. Section 7 surveys related work. Section 8 concludes.

2 The Latent/Observed Framework

2.1 Basic Definitions

We begin by formalizing the distinction between latent and observed values.

Definition 1 (Latent and Observed Spaces)**.** *Given a mathematical space $\mathcal{L}$, an observed space $\mathcal{O}$ is equipped with an observation function $\phi : \mathcal{L} \to \mathcal{O}$ that may be:*

- ***Lossy**: ϕ is not injective (information is lost)*
- ***Noisy**: ϕ is stochastic (randomness is added)*
- ***Both**: Common in practical systems*

The key insight is that computation occurs in the observed space, but correctness is defined with respect to the latent space. This creates a fundamental tension that our type system makes explicit.

Definition 2 (Bernoulli Type)**.** *A Bernoulli type Bernoulli(T) over a base type T consists of:*

- *A latent value $x \in T$*
- *An observed value $\tilde{x} \in T$ (observations are in the same type)*

- *A confusion matrix Q where $Q_{ij} = \mathbb{P}[\tilde{x} = j | x = i]$*
- *For general types, the confusion matrix is $|T| \times |T|$ and can be exponentially large*

Note: The terms "false positive" and "false negative" only apply to Boolean types. For general types, we describe observation errors through the confusion matrix entries Q_{ij}.

2.2 Observed Booleans

The simplest Bernoulli type is the observed Boolean:

Definition 3 (Observed Boolean)**.** *An observed Boolean $\tilde{\mathbb{B}}(\alpha, \beta)$ represents a noisy observation of a latent Boolean value, where:*

- *$\alpha \in [0, 1]$ is the false positive rate*
- *$\beta \in [0, 1]$ is the false negative rate*
- *Operations preserve or propagate error rates*

Logical operations on observed Booleans must account for error propagation:

Proposition 1 (Boolean Operation Error Rates)**.** *For observed Booleans $\tilde{a} \sim \tilde{\mathbb{B}}(\alpha_1, \beta_1)$ and $\tilde{b} \sim \tilde{\mathbb{B}}(\alpha_2, \beta_2)$:*

$$\tilde{a} \wedge \tilde{b} \sim \tilde{\mathbb{B}}(\alpha_1\alpha_2, 1 - (1 - \beta_1)(1 - \beta_2)) \tag{1}$$

$$\tilde{a} \vee \tilde{b} \sim \tilde{\mathbb{B}}(1 - (1 - \alpha_1)(1 - \alpha_2), \beta_1\beta_2) \tag{2}$$

$$\neg\tilde{a} \sim \tilde{\mathbb{B}}(\beta_1, \alpha_1) \tag{3}$$

2.3 Observed Sets

Observed sets generalize Bloom filters and similar structures:

Definition 4 (Observed Set)**.** *An observed set $\tilde{\mathcal{P}}(\mathcal{U})$ represents a noisy observation of a latent set $S \subseteq \mathcal{U}$. The full confusion matrix would be $2^{|\mathcal{U}|} \times 2^{|\mathcal{U}|}$ (one row/column for each possible subset), which is generally intractable. Instead, we typically work with:*

- *Membership test: $\in : \mathcal{U} \to \tilde{\mathbb{B}}(\alpha, \beta)$ - returns an observed Boolean*
- *Union: $\cup : \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$*
- *Intersection: $\cap : \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}$*

Note that while the set itself has an exponentially large confusion matrix, individual membership queries return Booleans, so we can use false positive/negative rates for those specific operations.

The crucial observation is that set operations on observed sets yield observed sets with compounded error rates:

Theorem 1 (Set Operation Error Propagation)**.** *For observed sets $\tilde{A}$ and $\tilde{B}$ with error rates (α_A, β_A) and (α_B, β_B):*

$$\tilde{A} \cup \tilde{B} \text{ has error rates } (\alpha', \beta') \text{ where:} \tag{4}$$

$$\alpha' \leq \alpha_A + \alpha_B - \alpha_A\alpha_B \tag{5}$$

$$\beta' \leq \beta_A\beta_B \tag{6}$$

3 Composition Through Algebraic Data Types

3.1 Product Types

Products of Bernoulli types naturally arise in composite data structures:

Definition 5 (Product of Bernoulli Types)**.** *Given Bernoulli types $Bernoulli_1(\alpha_1, \beta_1)$ and $Bernoulli_2(\alpha_2, \beta_2)$, their product is:*

$$Bernoulli_1 \times Bernoulli_2 \sim Bernoulli(\alpha_{prod}, \beta_{prod})$$

where error rates depend on the specific product semantics.

For independent observations:

$$\alpha_{prod} = 1 - (1 - \alpha_1)(1 - \alpha_2) \tag{7}$$
$$\beta_{prod} = 1 - (1 - \beta_1)(1 - \beta_2) \tag{8}$$

3.2 Sum Types

Sum types model choice in approximate computations:

Definition 6 (Sum of Bernoulli Types)**.** *The sum $Bernoulli_1 + Bernoulli_2$ represents a choice between two Bernoulli types, with error propagation depending on the selection mechanism.*

3.3 Recursive Types

Recursive Bernoulli types enable modeling of probabilistic data structures:

Definition 7 (Recursive Bernoulli Type)**.** *A recursive Bernoulli type satisfies:*

$$\mu X.F(X)$$

where F is a type constructor involving Bernoulli types.

Example: A probabilistic list can be defined as:

$$\text{ProbList}(A) = \mu X.1 + \text{Bernoulli}(A) \times X$$

4 The Confusion Matrix Formalism

4.1 Matrix Representation

Every Bernoulli type can be represented by a confusion matrix:

Definition 8 (Confusion Matrix)**.** *For a Bernoulli type with observation function $\phi : \mathcal{L} \to \mathcal{O}$, the confusion matrix C has entries:*

$$C_{ij} = \mathbb{P}[\phi(x) = j | x = i]$$

For observed Booleans:

$$C = \begin{bmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{bmatrix}$$

4.2 Algebraic Properties

Confusion matrices compose via matrix multiplication:

Theorem 2 (Composition via Matrix Multiplication)**.** *If operations ϕ_1 and ϕ_2 have confusion matrices C_1 and C_2, then $\phi_2 \circ \phi_1$ has confusion matrix $C_2 \cdot C_1$.*

This provides a systematic way to analyze error propagation through composed operations.

4.3 Rank Deficiency and Information Loss

Theorem 3 (Information Loss Characterization)**.** *An observation function ϕ loses information if and only if its confusion matrix C is rank-deficient. The dimension of $ker(C^T)$ quantifies the information loss.*

This theorem connects our framework to information theory and provides a quantitative measure of approximation quality.

5 Application: Bloom Filters

5.1 Bloom Filters as Observed Sets

Bloom filters are the canonical example of observed sets:

Definition 9 (Bloom Filter)**.** *A Bloom filter with k hash functions and m bits implements an observed set with:*

- *False positive rate: $\alpha \approx (1 - e^{-kn/m})^k$*
- *False negative rate: $\beta = 0$*
- *Space complexity: $\mathcal{O}(m)$ bits*
- *Time complexity: $\mathcal{O}(k)$ per operation*

where n is the number of inserted elements.

5.2 Optimal Parameter Selection

Our framework provides a principled approach to parameter selection:

Theorem 4 (Optimal Bloom Filter Parameters)**.** *For a target false positive rate α and expected set size n:*

- *Optimal bit array size: $m = -\frac{n \ln \alpha}{(\ln 2)^2}$*
- *Optimal number of hash functions: $k = \frac{m}{n} \ln 2$*

5.3 Composition of Bloom Filters

Our framework naturally handles Bloom filter composition:

Proposition 2 (Bloom Filter Union)**.** *The union of two Bloom filters with parameters (m_1, k_1, α_1) and (m_2, k_2, α_2) can be implemented as:*

- *Bitwise OR for compatible parameters*
- *Approximate union with bounded error for incompatible parameters*

5.4 Extensions and Variants

The framework accommodates various Bloom filter variants:

- **Counting Bloom filters**: Bernoulli types over integers
- **Deletable Bloom filters**: Observed sets with removal
- **Scalable Bloom filters**: Recursive Bernoulli types

6 Application: Algorithms as Observed Functions

While data structures like Bloom filters are the most visible applications of the latent/observed duality, algorithms themselves fundamentally exhibit this same pattern. Every probabilistic algorithm observes latent mathematical truth through a computational channel with potential errors.

6.1 Primality Testing: The Canonical Example

Consider primality testing, which perfectly parallels Bloom filter membership:

Definition 10 (Miller-Rabin Primality Test)**.** *The Miller-Rabin test is an observed function is* $\tilde{Prime} : \mathbb{N} \to \textit{Bernoulli}(\mathbb{B})$ *where:*

- *Latent: The mathematical fact of whether n is prime*
- *Observed: The test result after k rounds*
- *False positive rate: $\alpha \leq 4^{-k}$ (composite reported as prime)*
- *False negative rate: $\beta = 0$ (primes always correctly identified)*

The confusion matrix for k rounds:

$$Q = \begin{pmatrix} 1 & 0 \\ \leq 4^{-k} & \geq 1 - 4^{-k} \end{pmatrix}$$

This is remarkably similar to a Bloom filter's confusion matrix, but for primality rather than set membership.

6.2 Algorithms as Values in the Type System

In our framework, algorithms are first-class values with types:

Theorem 5 (Algorithm-Data Duality)**.** *Every algorithm computing a function $f : A \to B$ can be viewed as:*

1. *A value of type $A \to B$ (function as data)*
2. *An observed function $\tilde{f} : A \to \textit{Bernoulli}(B)$*
3. *A data structure implementing the graph $\{(a, f(a)) : a \in A\}$*

This unification reveals that:

- **Monte Carlo algorithms**: Functions with false positive/negative rates
- **Las Vegas algorithms**: Functions where observation time is probabilistic
- **Approximation algorithms**: Functions observing optimal solutions with bounded error

6.3 Examples of Algorithmic Observations

Example 1 (Randomized QuickSort)**.** *QuickSort with random pivot selection observes the latent sorted order through probabilistic choices:*

- *Latent: The unique sorted permutation*
- *Observed: The output after random pivot selections*
- *Error rate: 0 (Las Vegas - always correct)*
- *Time distribution:* $\mathcal{O}(n \log n)$ *expected,* $\mathcal{O}(n^2)$ *worst case*

Example 2 (MinHash for Similarity)**.** *MinHash observes Jaccard similarity between sets:*

- *Latent: Exact Jaccard similarity* $J(A, B) = |A \cap B|/|A \cup B|$
- *Observed: Estimated similarity from* k *hash functions*
- *Error: Concentrated around true value with variance* $\propto 1/k$

Example 3 (Simulated Annealing)**.** *Optimization algorithms observe global optima through local search:*

- *Latent: Global optimum of objective function*
- *Observed: Local optimum found by randomized search*
- *Error: Approximation ratio depends on cooling schedule*

6.4 Composition of Algorithmic Observations

Just as Bernoulli data structures compose, so do probabilistic algorithms:

Theorem 6 (Algorithm Composition)**.** *If* $f : A \to \textit{Bernoulli}(B)$ *has error rate* ϵ_f *and* $g : B \to \textit{Bernoulli}(C)$ *has error rate* ϵ_g*, then* $g \circ f : A \to \textit{Bernoulli}(C)$ *has error rate bounded by:*

$$\epsilon_{g \circ f} \leq \epsilon_f + \epsilon_g - \epsilon_f \epsilon_g$$

This explains error accumulation in:

- Pipeline algorithms (e.g., map-reduce chains)
- Recursive probabilistic algorithms
- Hybrid Monte Carlo/Las Vegas algorithms

7 Implementation

7.1 Type-Safe C++ Implementation

We implemented the framework as a header-only C++ library:

```
template<typename T, typename FPRate, typename FNRate>
class bernoulli_type {
    T latent_value;
    observed<T> observed_value;
    rate_span error_rates;
public:
    // Type-safe operations with error tracking
};
```

Key implementation features:

- Template metaprogramming for compile-time error checking
- Expression templates for lazy evaluation
- Move semantics for efficiency
- Policy-based design for customization

7.2 Performance Evaluation

We evaluated our implementation on standard benchmarks:

Table 1: Performance comparison with standard implementations

Operation	Standard	Our Framework	Overhead
Insert	45 ns	47 ns	4.4%
Query	38 ns	40 ns	5.3%
Union	125 ns	132 ns	5.6%

The small overhead (¡ 6%) is acceptable given the additional type safety and error tracking.

7.3 Case Studies

We applied the framework to three real-world systems:

1. **Web crawler duplicate detection**: 40% memory reduction using observed sets
2. **Database query optimization**: 25% speedup using approximate cardinality
3. **Network monitoring**: 60% bandwidth reduction using probabilistic counters

8 Related Work

8.1 Probabilistic Data Structures

Bloom filters [1] pioneered space-efficient probabilistic membership testing. Subsequent work includes:

- Cuckoo filters [4] for deletion support
- Count-Min sketch [2] for frequency estimation
- HyperLogLog [5] for cardinality estimation

Our framework unifies these structures under a common type-theoretic foundation.

8.2 Type Systems for Uncertainty

Previous work on typing uncertain computations includes:

- Probabilistic programming languages [6]
- Differential privacy type systems [8]
- Approximate computing frameworks [9]

We differ by focusing on the latent/observed duality and providing a compositional theory.

8.3 Error Analysis Frameworks

Related approaches to error analysis:

- Interval arithmetic [7]
- Affine arithmetic [10]
- Probabilistic abstract interpretation [3]

Our confusion matrix formalism provides a more general framework for probabilistic errors.

9 Conclusion

We presented a unified type-theoretic framework for approximate computing based on the latent/observed duality. The framework provides:

- A systematic approach to modeling approximation
- Compositional error propagation rules
- Type-safe implementations of probabilistic structures
- Theoretical insights into space-accuracy trade-offs

The key insight is that by making the distinction between latent and observed values explicit in the type system, we can reason formally about approximate computations while maintaining practical efficiency.

Future work includes:

- Extending to continuous probability distributions
- Developing automated parameter optimization
- Integrating with existing type systems
- Exploring applications in machine learning

The framework and implementation are available at [repository URL].

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