

Statistical Inference for Series Systems from Masked Failure Time Data: The Exponential Case

Alexander Towell
atowell@siue.edu

Abstract

We consider the problem of estimating component failure rates in series systems when observations consist of system failure times paired with partial information about the failed component. For the case where component lifetimes follow exponential distributions, we derive closed-form expressions for the maximum likelihood estimator, the Fisher information matrix, and establish minimal sufficient statistics. The asymptotic sampling distribution of the estimator is characterized and confidence intervals are provided. A detailed analysis of a three-component system demonstrates the theoretical results.

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1 Introduction

Series systems are fundamental in reliability engineering: the system fails whenever any single component fails. In many practical situations, the exact cause of system failure cannot be determined with certainty, but can be narrowed to a subset of components. This partial information is known as *masked* failure data. For example, a diagnostic system might isolate a failure to one of two circuit boards without determining which board actually failed.

The problem of statistical inference from masked system failure data has received considerable attention in the reliability literature. Usher and Hodgson [6] introduced maximum likelihood methods for estimating component reliability from masked system life-test data, focusing on series systems with exponentially distributed component lifetimes. Lin et al. [5] developed exact maximum likelihood estimation procedures for more general masking scenarios. The competing risks framework, where system failure can be attributed to multiple failure modes, provides a natural setting for masked data analysis [1]. More recent work has addressed interval-censored data [3] and Bayesian approaches to masked failure data [4].

Despite this body of work, closed-form analytical results remain limited, particularly for the Fisher information matrix and asymptotic properties of estimators under general masking patterns. Most existing methods rely on numerical optimization or EM algorithms without explicit characterization of estimator variance.

This paper focuses on series systems where component lifetimes follow exponential distributions. The exponential assumption is justified in several contexts: systems subject to random external shocks, systems with constant hazard rates, and as a tractable model for early-stage analysis. More importantly, the exponential case admits closed-form solutions that provide insight into the information structure of masked failure data.

1.1 Contributions

Our main contributions are:

1. Closed-form expression for the maximum likelihood estimator of component failure rates from masked system failure times
2. Explicit formula for the Fisher information matrix under various masking scenarios
3. Identification of minimal sufficient statistics (mean system lifetime and candidate set frequencies)
4. Characterization of the asymptotic sampling distribution and construction of confidence intervals
5. Detailed analysis of three-component systems including numerical validation

1.2 Paper organization

Section 2 introduces the probabilistic model for series systems with masked failures. Section 6 develops the likelihood and Fisher information for general parametric families. Section 9 presents the main results for exponentially distributed components, including the MLE, information matrix, and sufficient statistics. Section 15 provides detailed analysis of three-component systems. Section 19 concludes with discussion.

2 Probabilistic Model

3 Series System Lifetime

Consider a system composed of m components. Component j has a random lifetime $T_j > 0$ for $j = 1, \dots, m$. We make the following assumptions:

Assumption 3.1. The component lifetimes $T_1, \dots, T_m$ are mutually independent.

Assumption 3.2. The system operates if and only if all components are functioning (series configuration).

Under these assumptions, the system lifetime is:

$$S = \min(T_1, \dots, T_m) \quad (1)$$

Let $F_j(t)$ and $f_j(t)$ denote the CDF and PDF of component j . Define the reliability function $R_j(t) = 1 - F_j(t) = P\{T_j > t\}$.

The system reliability function is:

$$R_S(t) = \prod_{j=1}^m R_j(t) \quad (2)$$

The system PDF is:

$$f_S(t) = \sum_{j=1}^m \left(f_j(t) \prod_{\substack{k=1 \\ k \neq j}}^m R_k(t) \right) \quad (3)$$

4 Masked Component Failures

When the system fails at time t , exactly one component caused the failure. Let $K \in \{1, \dots, m\}$ denote the failed component. In many applications, K cannot be observed directly. Instead, we observe a *candidate set* $C \subseteq \{1, \dots, m\}$ that contains the failed component with some probability.

Definition 4.1. An α -masked component failure with cardinality w consists of a subset C of size w that contains the true failed component with probability α .

We assume:

Assumption 4.2. The masking accuracy α and cardinality w are independent of the system lifetime S and failed component K .

This assumption implies that the masking mechanism does not depend on which component failed or when, only on the diagnostic capabilities of the inspection process.

Definition 4.3. A *masked system failure time* is a triple (t, C, w) where t is the observed system failure time, C is the candidate set of cardinality w .

5 Parametric Families

We assume component j has a lifetime distribution from a parametric family indexed by parameter θ_j^* . The system parameter is $\Theta^* = (\theta_1^*, \dots, \theta_m^*)$.

Our data consists of a random sample of n independent masked system failure times:

$$\mathbf{M}_n = \{(t_1, C_1), \dots, (t_n, C_n)\} \quad (4)$$

where we condition on fixed cardinality w for simplicity.

6 Likelihood and Fisher Information

7 Likelihood Function

Given the model assumptions, the conditional probability that component k failed given system failure at time t is:

$$p_{K|S}(k|t, \Theta^*) = \frac{f_k(t) \prod_{j \neq k} R_j(t)}{f_S(t)} \quad (5)$$

Under the α -masking model with fixed cardinality w , the conditional probability of observing candidate set C given system failure at time t is:

$$p_{C|S,W}(C|t, w, \Theta^*) = \frac{\sum_{j \in C} f_j(t) \prod_{k \neq j} R_k(t)}{\binom{m-1}{w-1} f_S(t)} \quad (6)$$

The joint density of system failure time and candidate set is:

$$f_{C,S|W}(C, t|w, \Theta^*) = \frac{1}{\binom{m-1}{w-1}} \sum_{j \in C} f_j(t) \prod_{k \neq j} R_k(t) \quad (7)$$

The likelihood function for sample $\mathbf{M}_n$ is:

$$\mathcal{L}(\Theta|\mathbf{M}_n) = \prod_{i=1}^n f_{C,S|W}(C_i, t_i|w, \Theta) \quad (8)$$

8 Fisher Information Matrix

The Fisher information matrix quantifies the expected information about Θ^* contained in a single observation. Under regularity conditions, the (i, j) -th element is:

$$[\mathcal{I}(\Theta^*|w)]_{ij} = -E\left\{\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln f_{C,S|W}(C, S|w, \Theta)\right\}[\Theta^*] \quad (9)$$

For a sample of n observations, the information is additive:

$$\mathcal{I}_n(\Theta^*|w) = n \cdot \mathcal{I}(\Theta^*|w) \quad (10)$$

9 Exponentially Distributed Component Lifetimes

10 Exponential Parametric Functions

Suppose component j has an exponentially distributed lifetime with failure rate λ_j^* , denoted $T_j \sim \text{EXP}(\lambda_j^*)$. The component has:

$$R_j(t|\lambda_j^*) = \exp(-\lambda_j^* t) \quad (11)$$

$$f_j(t|\lambda_j^*) = \lambda_j^* \exp(-\lambda_j^* t) \quad (12)$$

$$h_j(t|\lambda_j^*) = \lambda_j^* \quad (13)$$

where $t > 0$ and $\lambda_j^* > 0$.

Theorem 10.1. *A series system with exponentially distributed component lifetimes is exponentially distributed with failure rate $\sum_{j=1}^m \lambda_j^*$.*

Proof. By equation (2),

$$R_S(t|\boldsymbol{\lambda}^*) = \prod_{j=1}^m \exp(-\lambda_j^* t) = \exp\left(-\left[\sum_{j=1}^m \lambda_j^*\right] t\right) \quad (14)$$

which is the reliability function of an exponential distribution with rate $\sum_{j=1}^m \lambda_j^*$. $\square$

The system PDF is:

$$f_S(t|\boldsymbol{\lambda}^*) = \left(\sum_{j=1}^m \lambda_j^*\right) \exp\left(-\left[\sum_{j=1}^m \lambda_j^*\right] t\right) \quad (15)$$

An important property of exponential series systems is that K and S are independent:

$$p_K(k|\boldsymbol{\lambda}^*) = \frac{\lambda_k^*}{\sum_{j=1}^m \lambda_j^*} \quad (16)$$

The joint density is:

$$f_{K,S}(k, t|\boldsymbol{\lambda}^*) = \lambda_k^* \exp\left(-\left[\sum_{j=1}^m \lambda_j^*\right] t\right) \quad (17)$$

Under the α -masking model, the joint density of candidate set and system failure time is:

$$f_{C,S|W}(C, t|w, \boldsymbol{\lambda}^*) = \frac{1}{\binom{m-1}{w-1}} \left(\sum_{j \in C} \lambda_j^* \right) \exp \left(- \left[\sum_{j=1}^m \lambda_j^* \right] t \right) \quad (18)$$

11 Maximum Likelihood Estimator

The likelihood function for sample $\mathbf{M}_n$ with candidate sets of cardinality w is:

$$\mathcal{L}(\boldsymbol{\lambda}|\mathbf{M}_n) \propto \exp \left(- \left[\sum_{j=1}^m \lambda_j \right] \left[\sum_{i=1}^n t_i \right] \right) \prod_{i=1}^n \left(\sum_{j \in C_i} \lambda_j \right) \quad (19)$$

The log-likelihood is:

$$\ell(\boldsymbol{\lambda}|\mathbf{M}_n) = \sum_{i=1}^n \ln \left(\sum_{j \in C_i} \lambda_j \right) - \left[\sum_{j=1}^m \lambda_j \right] \left[\sum_{i=1}^n t_i \right] \quad (20)$$

The score function has j -th component:

$$\frac{\partial \ell}{\partial \lambda_j} = \sum_{i=1}^n \frac{\mathbb{1}_{C_i}(j)}{\sum_{k \in C_i} \lambda_k} - \sum_{i=1}^n t_i \quad (21)$$

where $\mathbb{1}_C(j)$ is the indicator function equal to 1 if $j \in C$ and 0 otherwise.

Theorem 11.1. *The maximum likelihood estimator $\hat{\boldsymbol{\lambda}}_n$ maximizes the log-likelihood (20) and satisfies the score equation $\nabla \ell(\hat{\boldsymbol{\lambda}}_n|\mathbf{M}_n) = \mathbf{0}$.*

In general, this requires numerical solution. However, for specific cases (notably three-component systems with $w = 2$), closed-form solutions exist.

12 Sufficient Statistics

Theorem 12.1. *For masked system failure times from exponentially distributed series systems, the statistics*

$$\bar{t} = \frac{1}{n} \sum_{i=1}^n t_i \quad \text{and} \quad \hat{\boldsymbol{\omega}} = \{\hat{\boldsymbol{\omega}}_C : C \in [\{1, \dots, m\}]^w\} \quad (22)$$

where $\hat{\boldsymbol{\omega}}_C = \sum_{i=1}^n \mathbb{1}_C(C_i)$ are jointly sufficient for $\boldsymbol{\lambda}^*$.

Proof. The likelihood can be factored as:

$$\mathcal{L}(\boldsymbol{\lambda}|\mathbf{M}_n) = \exp\left(-n\bar{t}\sum_{j=1}^m\lambda_j\right)\prod_C\left(\sum_{j\in C}\lambda_j\right)^{\hat{\omega}_C} \quad (23)$$

which depends on the data only through $\bar{t}$ and $\hat{\omega}$. By the factorization theorem, these are sufficient statistics. $\square$

The sufficiency result shows that all information about $\boldsymbol{\lambda}^*$ in the sample is captured by: (1) the average system lifetime, and (2) the frequencies of each candidate set.

13 Fisher Information Matrix

The (j, k) -th element of the Fisher information matrix for exponential series systems is:

$$[\mathcal{I}(\boldsymbol{\lambda}^*|w)]_{jk} = \frac{\sum_C \left(\sum_{p\in C} \lambda_p^*\right)^{-1} \mathbb{1}_{C\times C}(j, k)}{\binom{m-1}{w-1} \sum_{p=1}^m \lambda_p^*} \quad (24)$$

where the sum is over all candidate sets C of cardinality w .

Proof. From equation (9), we compute:

$$[\mathcal{I}(\boldsymbol{\lambda}^*|w)]_{jk} = -\sum_C \int_0^\infty \frac{\partial^2}{\partial \lambda_j \partial \lambda_k} \ln f_{C,S|W}(C, t|w, \boldsymbol{\lambda}) \Big|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}^*}. \quad (25)$$

$$f_{C,S|W}(C, t|w, \boldsymbol{\lambda}^*) dt \quad (26)$$

The second derivative of the log density is:

$$\frac{\partial^2}{\partial \lambda_j \partial \lambda_k} \ln f_{C,S|W} = -\left(\sum_{p\in C} \lambda_p\right)^{-2} \mathbb{1}_{C\times C}(j, k) \quad (27)$$

Substituting the joint density (18) and integrating over t yields the result. $\square$

14 Asymptotic Sampling Distribution

Under regularity conditions, the MLE is consistent and asymptotically normal:

Theorem 14.1. *As $n \rightarrow \infty$,*

$$\sqrt{n}(\hat{\boldsymbol{\lambda}}_n - \boldsymbol{\lambda}^*) \xrightarrow{d} \text{MVN}(\mathbf{0}, \mathcal{I}^{-1}(\boldsymbol{\lambda}^*|w)) \quad (28)$$

This result follows from standard maximum likelihood theory [2]. The asymptotic variance-covariance matrix is the inverse of the Fisher information matrix.

14.1 Confidence Intervals

An asymptotic $(1 - \alpha) \times 100\%$ confidence interval for λ_j^* is:

$$\hat{\lambda}_j \pm z_{1-\alpha/2} \sqrt{\frac{1}{n} [\mathcal{I}^{-1}(\hat{\boldsymbol{\lambda}}_n|w)]_{jj}} \quad (29)$$

where $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$ -quantile of the standard normal distribution.

15 Three-Component Systems

We provide detailed analysis for systems with $m = 3$ components, which admits several closed-form results.

16 Candidate Sets of Size Two

Consider observations where each candidate set has cardinality $w = 2$. The three possible candidate sets are $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$.

The log-likelihood is:

$$\ell(\boldsymbol{\lambda}|\bar{t}, \hat{\boldsymbol{\omega}}) = \hat{\boldsymbol{\omega}}_{\{1,2\}} \ln(\lambda_1 + \lambda_2) + \hat{\boldsymbol{\omega}}_{\{1,3\}} \ln(\lambda_1 + \lambda_3) + \hat{\boldsymbol{\omega}}_{\{2,3\}} \ln(\lambda_2 + \lambda_3) - n\bar{t}(\lambda_1 + \lambda_2 + \lambda_3) \quad (30)$$

The score equations are:

$$\nabla \ell = \begin{pmatrix} \frac{\hat{\boldsymbol{\omega}}_{\{1,2\}}}{\lambda_1 + \lambda_2} + \frac{\hat{\boldsymbol{\omega}}_{\{1,3\}}}{\lambda_1 + \lambda_3} \\ \frac{\hat{\boldsymbol{\omega}}_{\{1,2\}}}{\lambda_1 + \lambda_2} + \frac{\hat{\boldsymbol{\omega}}_{\{2,3\}}}{\lambda_2 + \lambda_3} \\ \frac{\hat{\boldsymbol{\omega}}_{\{1,3\}}}{\lambda_1 + \lambda_3} + \frac{\hat{\boldsymbol{\omega}}_{\{2,3\}}}{\lambda_2 + \lambda_3} \end{pmatrix} - n\bar{t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0} \quad (31)$$

Theorem 16.1. *For three-component systems with $w = 2$, the MLE has the closed-form solution:*

$$\hat{\lambda}_n = \frac{1}{nt} \begin{pmatrix} \hat{\omega}_{\{1,2\}} + \hat{\omega}_{\{1,3\}} - \hat{\omega}_{\{2,3\}} \\ \hat{\omega}_{\{1,2\}} - \hat{\omega}_{\{1,3\}} + \hat{\omega}_{\{2,3\}} \\ -\hat{\omega}_{\{1,2\}} + \hat{\omega}_{\{1,3\}} + \hat{\omega}_{\{2,3\}} \end{pmatrix} \quad (32)$$

Proof. Setting the score to zero and solving the linear system yields the result. $\square$

The Fisher information matrix is:

$$\mathcal{I}(\lambda^*|w=2) = \frac{1}{2(\lambda_1^* + \lambda_2^* + \lambda_3^*)} \begin{bmatrix} \frac{1}{\lambda_1^* + \lambda_2^*} + \frac{1}{\lambda_1^* + \lambda_3^*} & \frac{1}{\lambda_1^* + \lambda_2^*} & \frac{1}{\lambda_1^* + \lambda_3^*} \\ \frac{1}{\lambda_1^* + \lambda_2^*} & \frac{1}{\lambda_1^* + \lambda_2^*} + \frac{1}{\lambda_2^* + \lambda_3^*} & \frac{1}{\lambda_2^* + \lambda_3^*} \\ \frac{1}{\lambda_1^* + \lambda_3^*} & \frac{1}{\lambda_2^* + \lambda_3^*} & \frac{1}{\lambda_1^* + \lambda_3^*} + \frac{1}{\lambda_2^* + \lambda_3^*} \end{bmatrix} \quad (33)$$

The inverse (asymptotic variance-covariance) is:

$$\mathcal{I}^{-1}(\lambda^*|w=2) = \frac{\lambda_1^* + \lambda_2^* + \lambda_3^*}{1} \begin{bmatrix} \lambda_1^* + \lambda_2^* + \lambda_3^* & -\lambda_3^* & -\lambda_2^* \\ -\lambda_3^* & \lambda_1^* + \lambda_2^* + \lambda_3^* & -\lambda_1^* \\ -\lambda_2^* & -\lambda_1^* & \lambda_1^* + \lambda_2^* + \lambda_3^* \end{bmatrix} \quad (34)$$

The asymptotic mean squared error (trace of variance-covariance) is:

$$\text{MSE}(\hat{\lambda}_n) = \frac{3(\lambda_1^* + \lambda_2^* + \lambda_3^*)^2}{n} \quad (35)$$

17 Candidate Sets of Size One

When $w = 1$, each observation identifies the exact failed component. This represents the no-masking case. The MLE is:

$$\hat{\lambda}_n = \frac{1}{nt} \begin{pmatrix} \hat{\omega}_{\{1\}} \\ \hat{\omega}_{\{2\}} \\ \hat{\omega}_{\{3\}} \end{pmatrix} \quad (36)$$

The Fisher information matrix is diagonal:

$$\mathcal{I}(\lambda^*|w=1) = \frac{1}{\lambda_1^* + \lambda_2^* + \lambda_3^*} \text{diag} \left(\frac{1}{\lambda_1^*}, \frac{1}{\lambda_2^*}, \frac{1}{\lambda_3^*} \right) \quad (37)$$

The asymptotic variance-covariance is:

$$\mathcal{I}^{-1}(\boldsymbol{\lambda}^*|w=1) = (\lambda_1^* + \lambda_2^* + \lambda_3^*)\text{diag}(\lambda_1^*, \lambda_2^*, \lambda_3^*) \quad (38)$$

The MSE is:

$$\text{MSE}(\hat{\boldsymbol{\lambda}}_n|w=1) = \frac{(\lambda_1^* + \lambda_2^* + \lambda_3^*)^2}{n} \quad (39)$$

which is exactly 1/3 the MSE when $w = 2$, reflecting the additional information from exact component identification.

18 Numerical Validation

We validate the asymptotic theory through simulation. Let $\boldsymbol{\lambda}^* = (2, 3, 4)^\top$ and $w = 2$. We generated $r = 10000$ samples of size $n = 1000$ and computed the MLE for each.

The theoretical asymptotic variance-covariance matrix (evaluated at $n = 1000$) is:

$$\frac{1}{1000}\mathcal{I}^{-1}(\boldsymbol{\lambda}^*|w=2) = \begin{bmatrix} 0.081 & -0.036 & -0.027 \\ -0.036 & 0.081 & -0.018 \\ -0.027 & -0.018 & 0.081 \end{bmatrix} \quad (40)$$

The sample variance-covariance from the 10000 MLEs is:

$$\widehat{\text{Cov}} = \begin{bmatrix} 0.081 & -0.037 & -0.027 \\ -0.037 & 0.082 & -0.018 \\ -0.027 & -0.018 & 0.081 \end{bmatrix} \quad (41)$$

The close agreement confirms the asymptotic approximation is accurate for $n = 1000$.

19 Conclusion

We have developed a comprehensive framework for statistical inference in series systems with exponentially distributed component lifetimes when failure data is masked. The exponential case admits closed-form expressions for the maximum likelihood estimator, Fisher information matrix, and sufficient statistics.

The main practical insights are:

1. The information content of masked data is quantified by the Fisher information matrix, which depends on the masking cardinality w
2. Minimal sufficient statistics are the mean system lifetime and candidate set frequencies
3. For three-component systems with $w = 2$, a closed-form MLE exists
4. Asymptotic confidence intervals provide practical uncertainty quantification

19.1 Extensions

Several extensions merit investigation:

1. Variable masking cardinality across observations
2. Non-exponential lifetime distributions (Weibull, gamma)
3. Covariate effects on failure rates
4. Bayesian approaches when prior information is available
5. Optimal inspection strategies to minimize expected masking

The exponential case provides a foundation for understanding masked system data. While the exponential assumption is restrictive, the analytical tractability enables clear insight into the information structure that extends conceptually to more complex settings.

A Numerical Solution Methods

For cases without closed-form solutions, the MLE must be computed numerically. The Newton-Raphson algorithm is effective:

Algorithm 1: Newton-Raphson for Exponential MLE

Input: Initial guess $\boldsymbol{\lambda}^{(0)}$, tolerance ϵ

Output: MLE $\hat{\boldsymbol{\lambda}}_n$

```
1  $k \leftarrow 0$ ;  
2 repeat  
3   Compute score  $\mathbf{s}^{(k)} = \nabla \ell(\boldsymbol{\lambda}^{(k)} | \mathbf{M}_n)$ ;  
4   Compute Hessian  $\mathbf{H}^{(k)} = \nabla^2 \ell(\boldsymbol{\lambda}^{(k)} | \mathbf{M}_n)$ ;  
5   Update  $\boldsymbol{\lambda}^{(k+1)} = \boldsymbol{\lambda}^{(k)} - [\mathbf{H}^{(k)}]^{-1} \mathbf{s}^{(k)}$ ;  
6    $k \leftarrow k + 1$ ;  
7 until  $\|\boldsymbol{\lambda}^{(k+1)} - \boldsymbol{\lambda}^{(k)}\| < \epsilon$ ;  
8 return  $\boldsymbol{\lambda}^{(k)}$ 
```

The Hessian for the exponential log-likelihood is:

$$[\nabla^2 \ell(\boldsymbol{\lambda} | \mathbf{M}_n)]_{jk} = - \sum_{i=1}^n \frac{\mathbb{1}_{C_i \times C_i}(j, k)}{\left(\sum_{p \in C_i} \lambda_p \right)^2} \quad (42)$$

Convergence is typically rapid when initialized at a reasonable starting point such as $\boldsymbol{\lambda}^{(0)} = (1/\bar{t}, \dots, 1/\bar{t})$.

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